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Exercise 2.2.19

$\langle x_n \rangle_n$ is a sequence of strictly positive $\mathbb{R}$. Prove that $\lim_n x_n = \infty$ iff $\lim_n \frac{1}{x_n} = 0$.

Prove $x_n \to \infty$ if $\frac{1}{x_n} \to 0$:

Assume $\frac{1}{x_n} \to 0$

then:

$$(\forall \varepsilon > 0)(\exists N)(\forall n \geq N)\left[\left|\frac{1}{x_n}\right| < \varepsilon\right] \quad \text{(assumption)}$$

so $\left|\frac{1}{x_n}\right| < \varepsilon \quad (\forall n \geq N)(\forall \varepsilon > 0)$

$\frac{1}{x_n} < \varepsilon$ (x_n is strictly positive)

then $x_n > \frac{1}{\varepsilon}$ (ordered field)

So, if we let $K = \frac{1}{\varepsilon}$

$$(\forall K > 0)(\exists N)(\forall n \geq N)[\,|x_n| > K\,]$$

and hence, x_n is a convergent sequence with limit ∞.

Prove $\frac{1}{x_n} \to 0$ if $x_n \to \infty$

$$(\forall K>0)(\exists N)(\forall n \geq N)[\,|x_n| > K\,] \quad \text{(Assumption)}$$

so for $(\forall K>0)(\forall n \geq N)$ for a qualifying N:

$$|x_n| > K$$

then $\quad x_n > K \quad$ (x_n is strictly positive)

$$\frac{1}{x_n} < \frac{1}{K}$$

If we let $\frac{1}{K} = \varepsilon$, we have

$$(\forall \varepsilon>0)(\exists N)(\forall n \geq N)\left[\,\left|\frac{1}{x_n}\right| < \varepsilon\,\right]$$

And hence $\frac{1}{x_n}$ is a convergent sequence with a limit of 0.